STRAND H: Angle Geometry

H3 *Constructions and Loci*

Text

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H3 Constructions and Loci

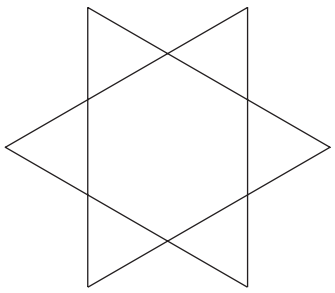
H3.1 Drawing and Symmetry

This section revises the ideas of symmetry first introduced in Unit H1 and gives you practice in drawing simple shapes.

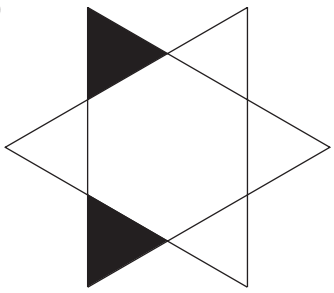
Worked Example 1

Describe the symmetries of each shape below.

(a)

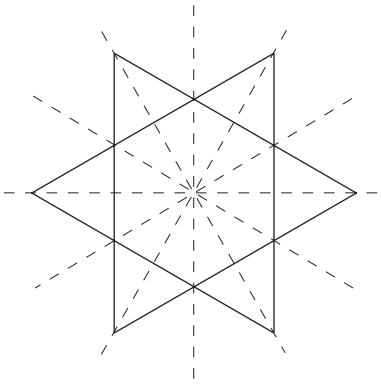


(b)



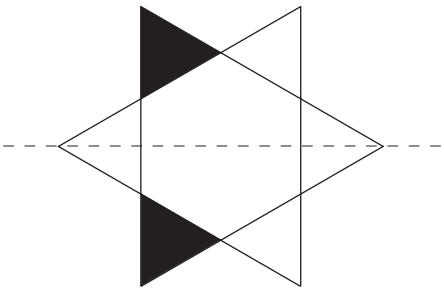
Solution

(a) This shape has 6 lines of symmetry, as shown in the diagram.



It has rotational symmetry of order 6 as it can be rotated about its centre to 6 different positions.

(b) This shape has one line of symmetry as shown below.



It has rotational symmetry of order 1, since it can rotate a full 360° back to its original position.

Worked Example 2

Draw accurately a rectangle with sides of length 8 cm and 5 cm.

Solution

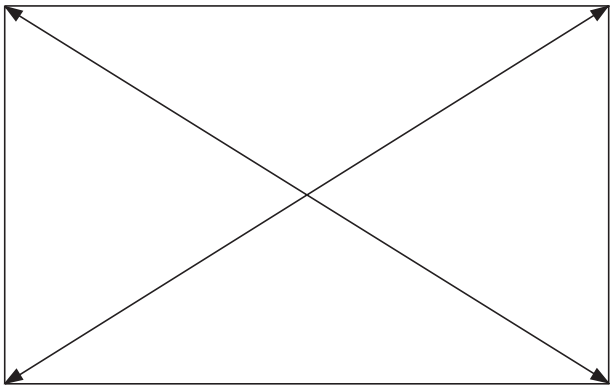
First draw a line 8 cm long:.

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Then draw lines 5 cm long at each end, making sure they are at right angles to the base line.



Finally, join these two lines to complete the rectangle.



Measure the diagonals and check that they are both the same length.

Exercises

1. Draw accurately rectangles with the following sizes.

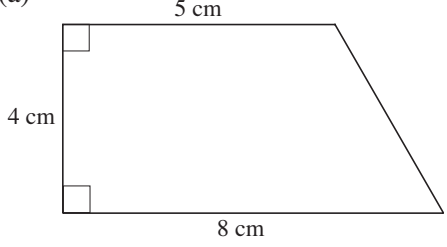
   (a) 3 cm by 8 cm (b) 10 cm by 3 cm

   (c) 6 cm by 7 cm (d) 6 cm by 4 cm

   For each rectangle check that both diagonals are the same length.

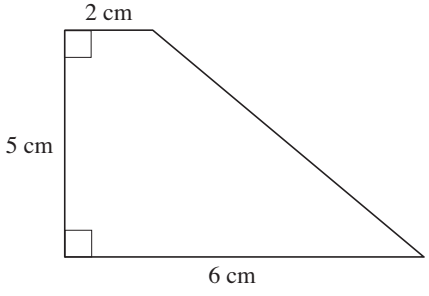
2. Make accurate drawings of each of the shapes shown below and answer the question below each shape.

   (a)



   What is the length of the sloping side?

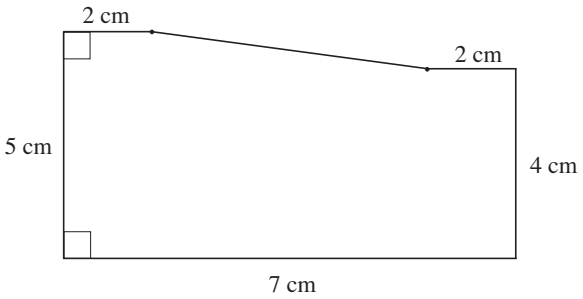
   (b)



   What is the length of the longest diagonal?

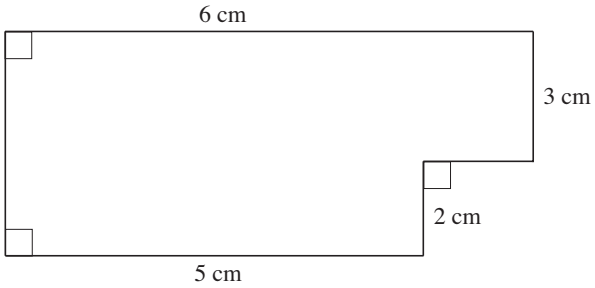
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(c)



What is the length of the sloping side?

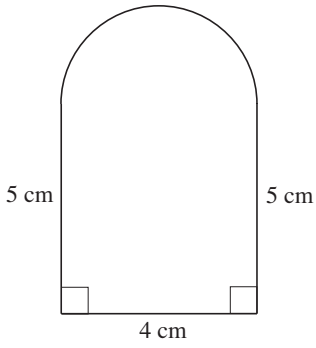
(d)



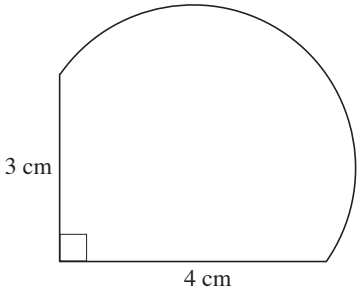
What is the length of the longest straight line which can be drawn inside the shape?

3. Each shape below includes a semi-circle. Make an accurate drawing of each shape and state the radius of the semi-circle.

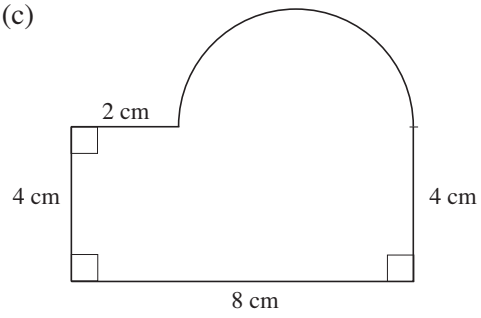
(a)



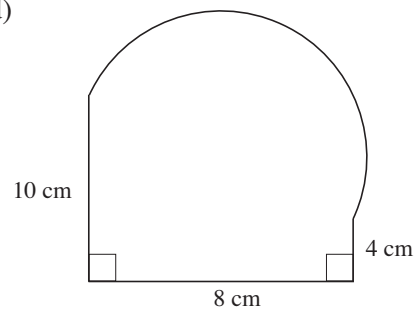
(b)



(c)



(d)

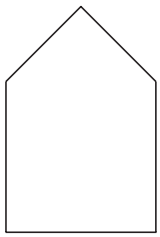


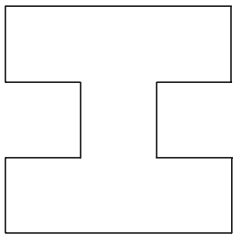
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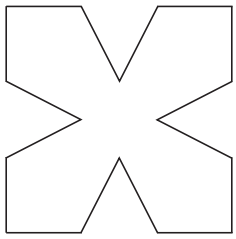
4. For each shape below:

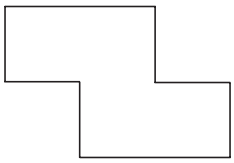
   (i) state the order of rotational symmetry,

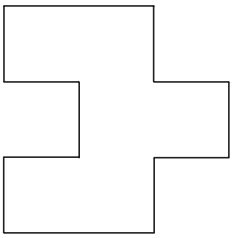
   (ii) copy the shape and draw any line of symmetry.

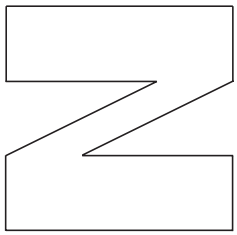
(a) 

(b) 

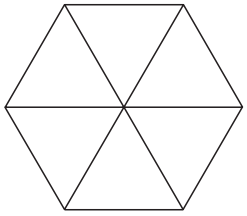
(c) 

(d) 

(e) 

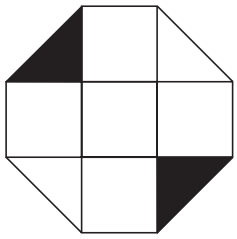
(f) 

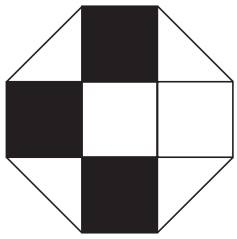
5. (a) Copy and shade part of the shape below so that it has 3 lines of symmetry.

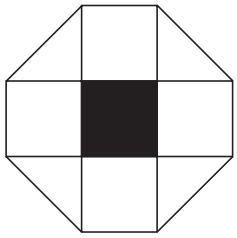


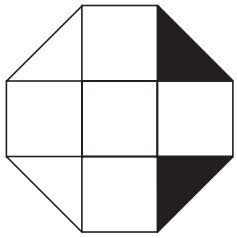
   (b) What is the order of rotational symmetry of the shape?

6. State the number of lines of symmetry and the order of rotational symmetry for each shape below.

(a) 

(b) 

(c) 

(d) 

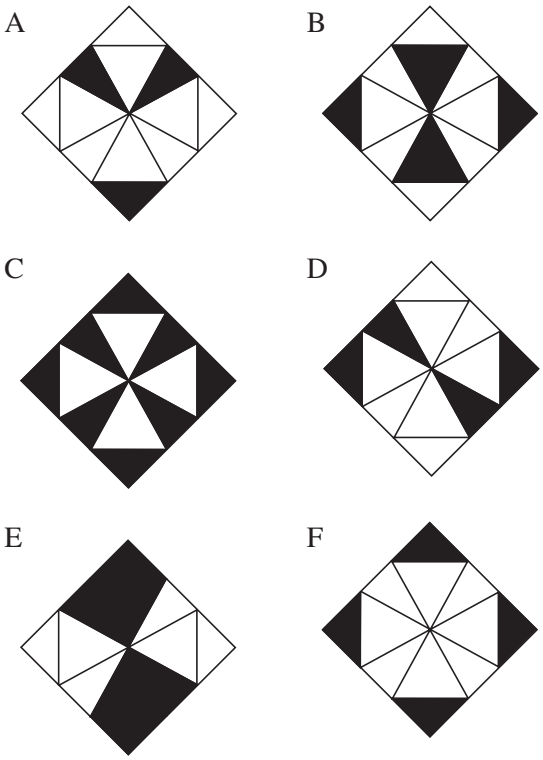
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7. Which of the shapes below have:

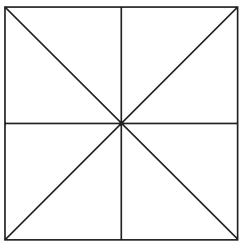
(a) rotational symmetry of order 1 (b) no lines of symmetry

(c) more than two lines of symmetry (d) rotational symmetry of order 2

(e) rotational symmetry of an order greater than 2?



8. Make 4 copies of the shape below.



Shade triangles in the shape to produce shapes with:

(a) 2 lines of symmetry (b) one line of symmetry

(c) rotational symmetry of order 2, (d) rotational symmetry of order 4.

Investigation

Look at an atlas and find out the scales used in maps of different countries. Are the same scales used for all the maps? If not, why not?

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H3.2 Constructing Triangles and Other Shapes

A protractor and a compass can be used to produce accurate drawings of triangles and other shapes.

We first recap some basic constructions that you will have met before.

Worked Example 1

Construct the perpendicular bisector of the line AB.

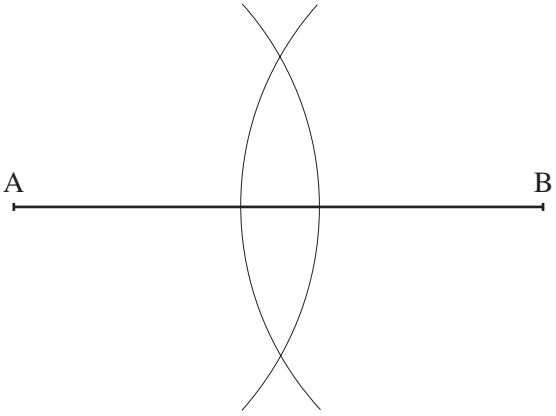
Then label the midpoint of AB, M.



Solution

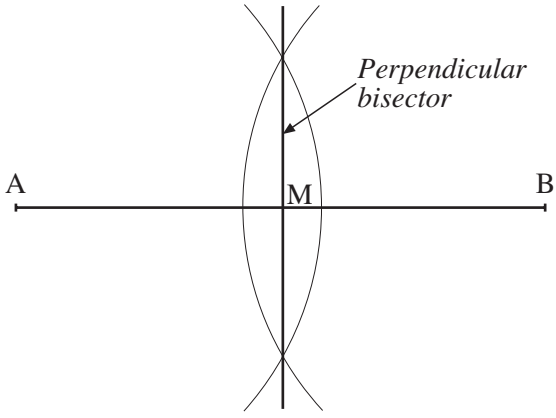
There are many lines that cut AB exactly in half. We have to construct the one that is perpendicular to AB.

We begin by drawing arcs of equal radius, centred on the points A and B, as shown in the diagram.
The radius of these arcs should be roughly $\frac{2}{3}$ to $\frac{3}{4}$ of the length AB.



Then draw a line through the intersection points of the two arcs.

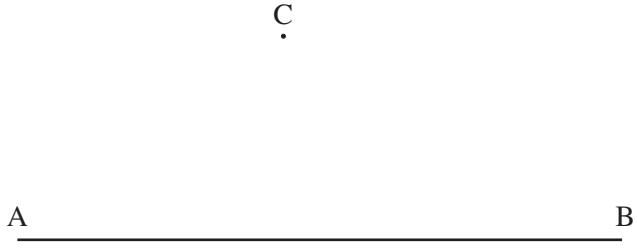
The point where the bisector intersects AB can then be labelled M.



Worked Example 2

The diagram shows the line AB and the point C.

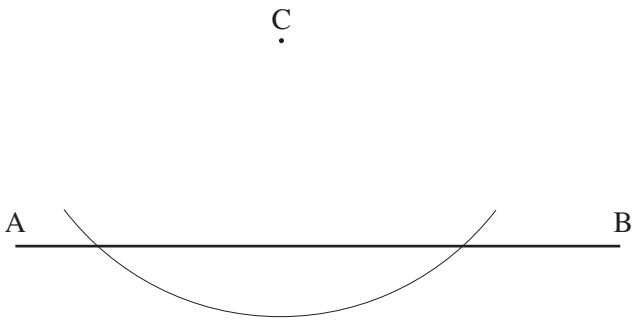
Draw a line through C that is perpendicular to AB.



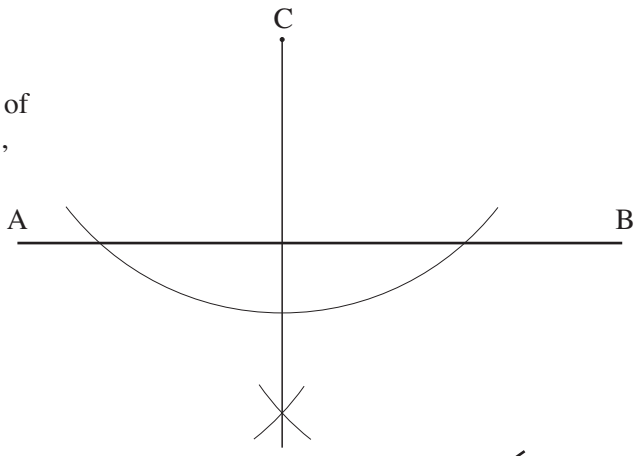
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Solution

Using C as the centre, draw an arc as shown.

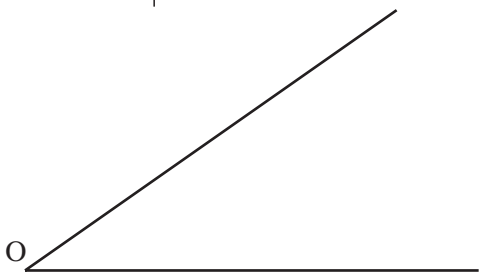


Then using the intersection points of this arc with the line AB as centres, draw two further arcs with radii of equal length. The perpendicular line can then be drawn from C through the point where these two new arcs cross.



Worked Example 3

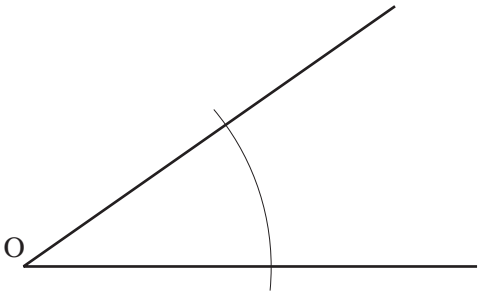
Bisect this angle.



Solution

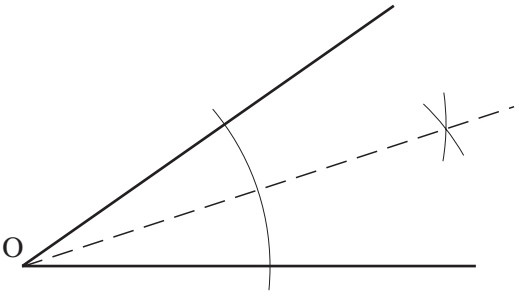
To bisect an angle you need to draw a line that cuts the angle in half.

First draw an arc using O as the centre.



Then draw two further arcs of equal radius, using the points where the arc intersects the lines as the centres.

The bisector can then be drawn from O through the point where these two new arcs cross.



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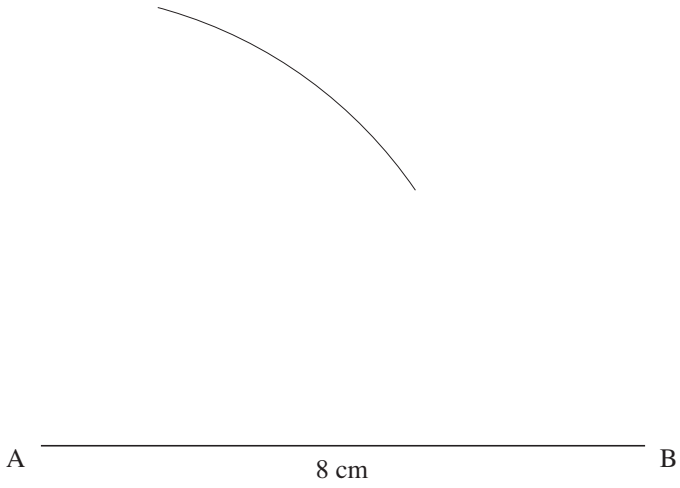
Worked Example 4

Construct a triangle with sides of length 8 cm, 6 cm and 6 cm.

Solution

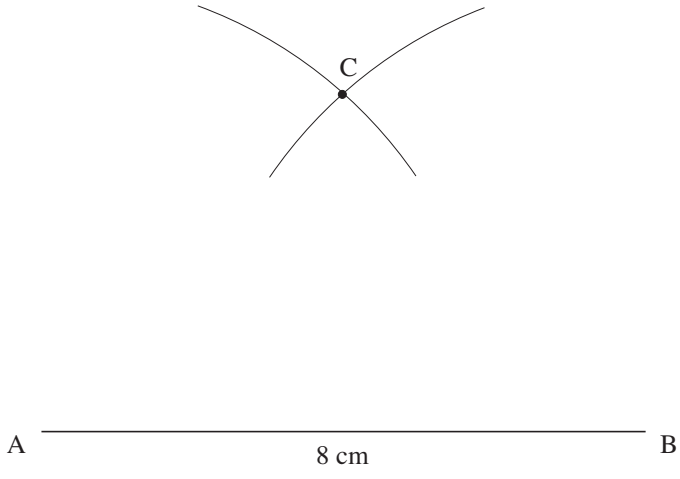
First draw a line of length 8 cm.

Then set the distance between the point and pencil of your compass to 6 cm and draw an arc with centre A as shown below.



The arc is a distance of 6 cm from A.

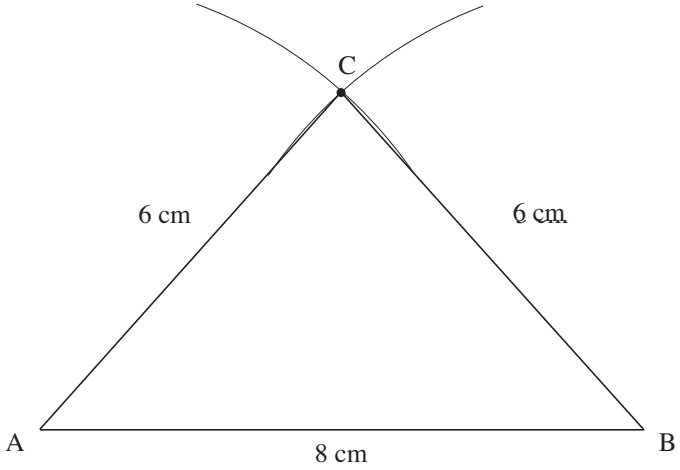
With your compass set so that the distance between the point and the pencil is still 6 cm, draw an arc centred at B, as shown below.



The point, C, where the two arcs intersect is the third vertex of the triangle.

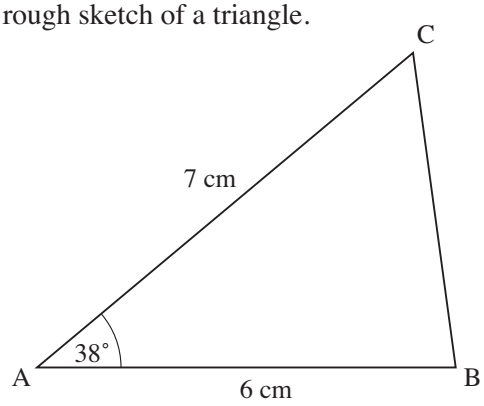
The triangle can now be completed.

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Worked Example 5

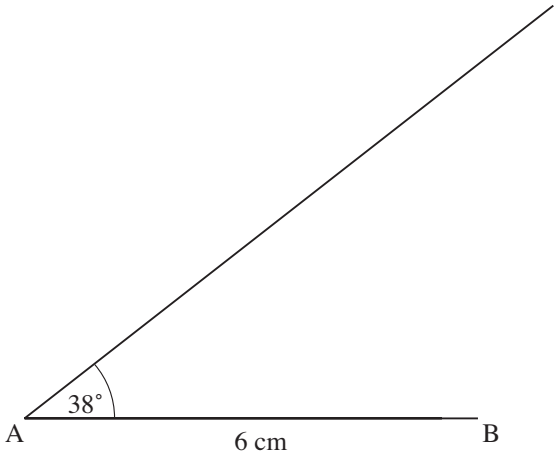
The diagram shows a rough sketch of a triangle.



Make an accurate drawing of the triangle, using a ruler and protractor, and find the length of the third side.

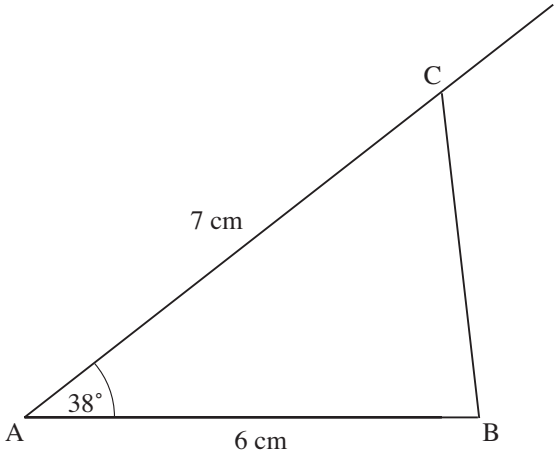
Solution

First draw a line of length 6 cm and measure an angle of $38°$.



Then measure 7 cm along the line and the triangle can be completed.

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The third side of the triangle can then be measured as approximately 4.3 cm.

Worked Example 6

(a) Draw a line segment, *PQ,* 7 cm long.

(b) **Using only a ruler, a pencil and a pair of compasses**, construct a line segment, *LM,* the perpendicular bisector of *PQ,* such that

LM cuts *PQ* at *O,* and $OL = OM = 4$ cm.

(c) Form parallelogram *PLQM* by joining the points *P, L, Q* and *M*.

(d) Measure and state the size of the angle *MPL*.

(e) What type of parallelogram is *PLQM*? Give a reason for your answer.

Solution

(a)

P 7 cm *Q*

(b) With compass centre *P* and centre *Q* and radius, say 7 cm, draw circle segments. The points of intersection form the perpendicular to *PQ,* crossing at point *O*. Use a ruler to find points *L* and *M* on this line.

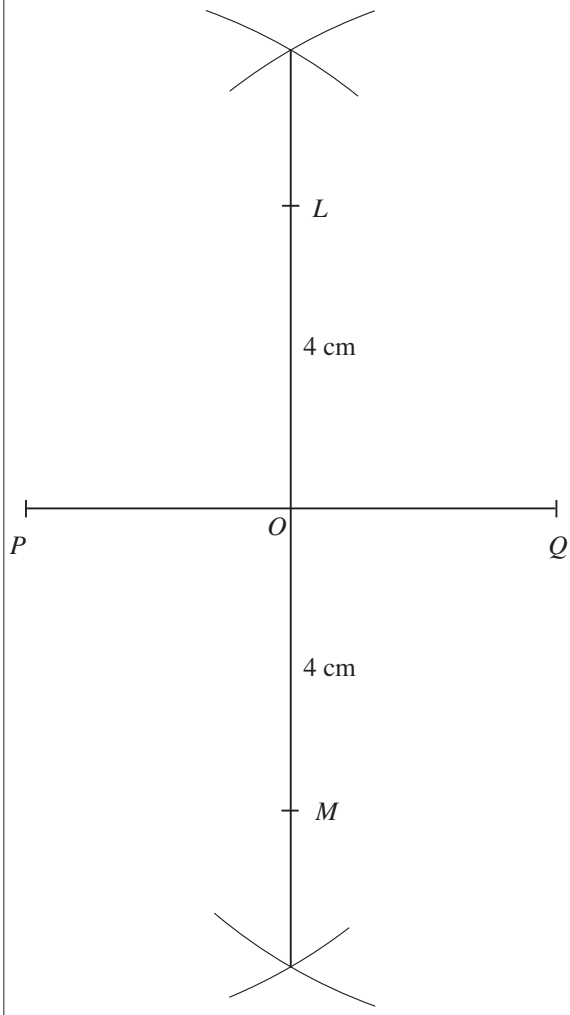
(c) See diagram on next page.

(d) Angle $MPL \approx 98°$

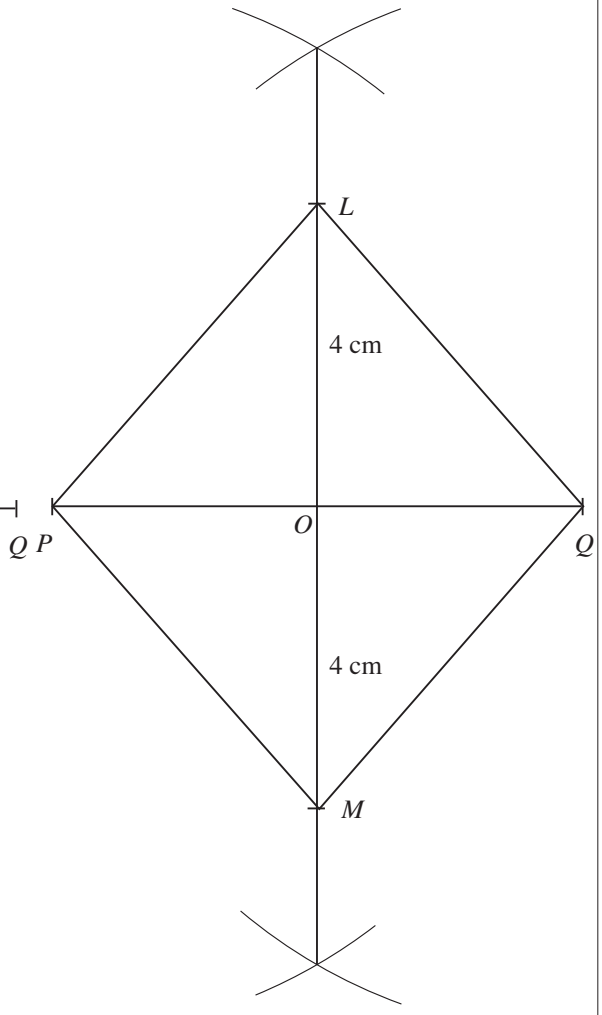
(e) *PLQM* is a rhombus as all sides are equal and angles are not equal to $90°$.

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(b)

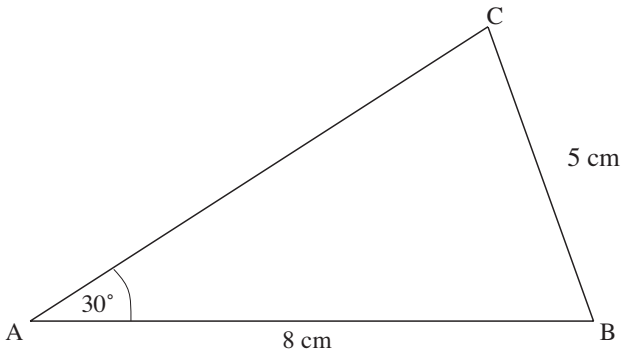


(c)



Worked Example 7

The diagram shows a rough sketch of a triangle.



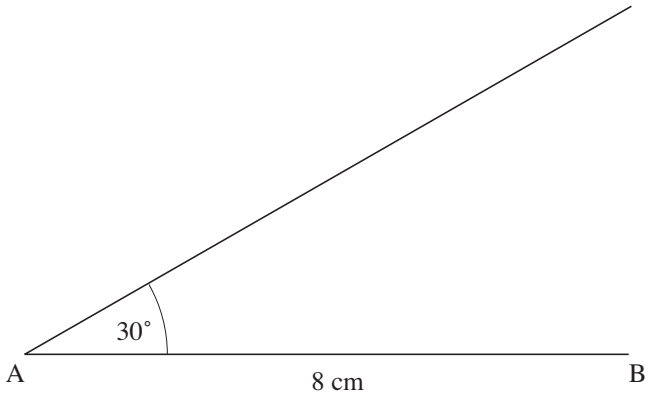
Make an accurate drawing of the triangle.

What is the length AC?

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Solution

First draw the side of length 8 cm and measure the angle of $30°$, using a protractor, as shown below.



[Alternatively, you can construct an angle of $60°$ and then bisect it to obtain an angle of $30°$.

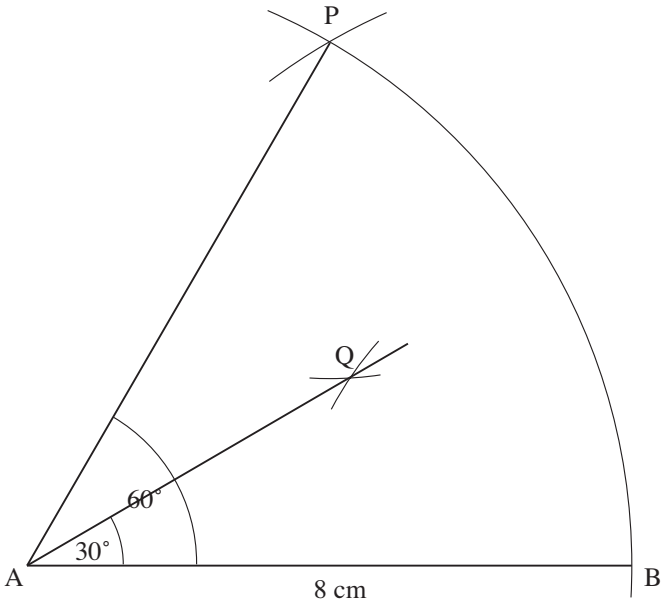
To construct an angle of $60°$:

Draw the line AB of length 8 cm.

Using compasses, draw a sector of the circle, centre A, radius 8 cm, as shown below.

With your compass point placed at point B, mark off the intersection of this circle with a second circle of radius 8 cm, centre B. Call this point P.

Join A to P. Angle PAB is $60°$.



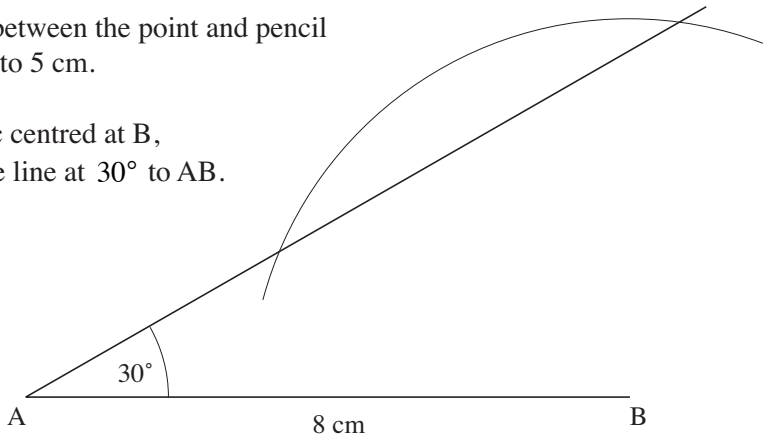
You can obtain an angle of $30°$ by bisecting angle PAB in the usual way (see diagram), by drawing sectors of circles, radii approximately 4.5 cm, centred on points P and B, marking the point of intersection as Q.

Join A to Q. Angle QAB is $30°$.]

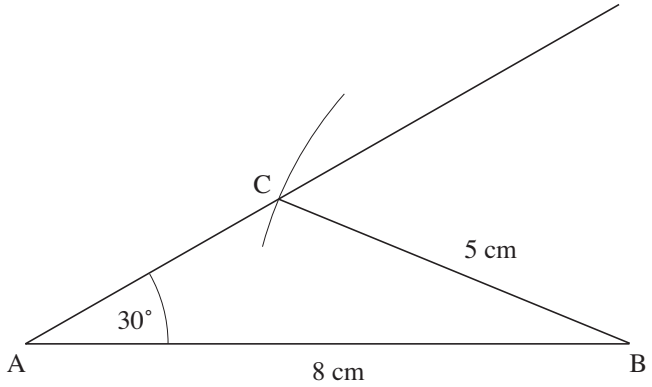
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Set the distance between the point and pencil of your compass to 5 cm.

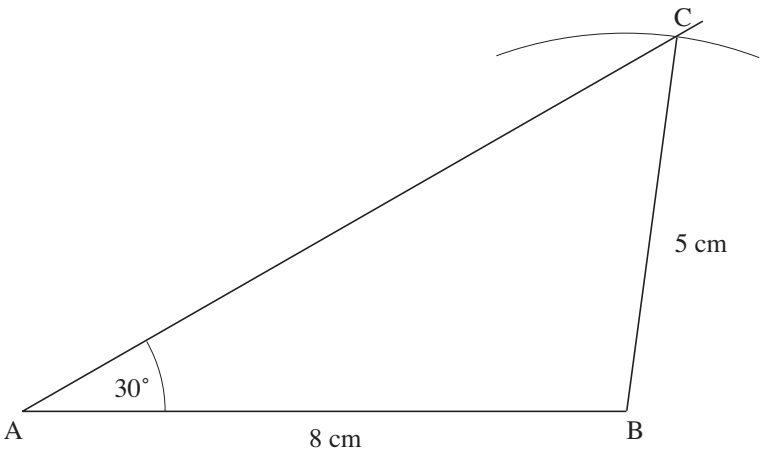
Then draw an arc centred at B, which crosses the line at 30° to AB.



As the arc crosses the line in two places, there are two possible triangles that can be constructed as shown below.



Both triangles have the lengths and angle specified in the rough sketch.



The possible lengths of AC are, approximately, 3.5 cm and 9 cm.

Note

An arc must be taken when constructing triangles to ensure that all possibilities are considered.

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Exercises

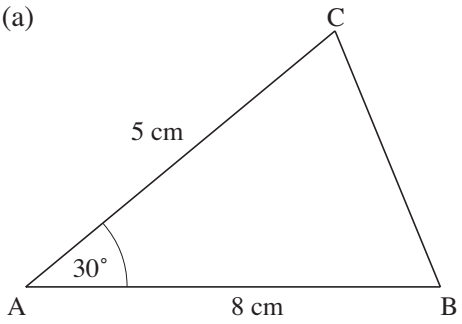
1. Draw triangles with sides of the following lengths.

   (a) 10 cm, 6 cm, 7 cm (b) 5 cm, 3 cm, 6 cm

   (c) 4 cm, 7 cm, 6 cm

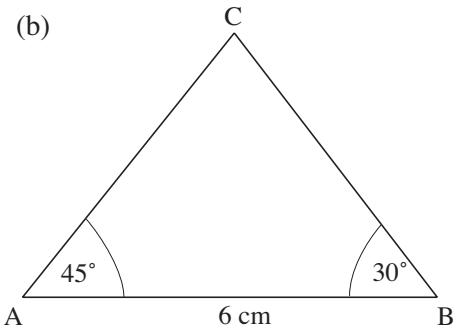
2. Draw accurately the triangles shown in the rough sketches below and answer the question given below each sketch.

(a)



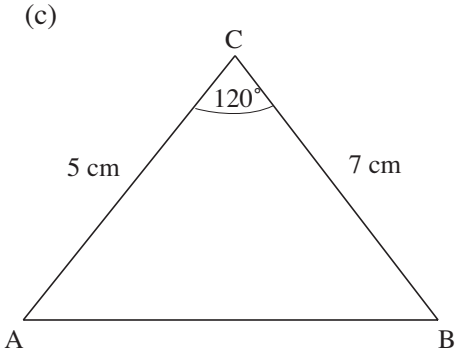
How long is the side BC?

(b)



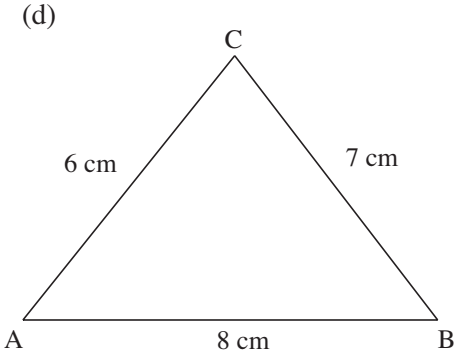
How long are the sides AC and BC?

(c)



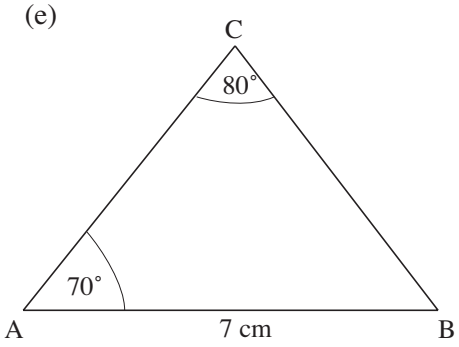
How long is the side AB?

(d)



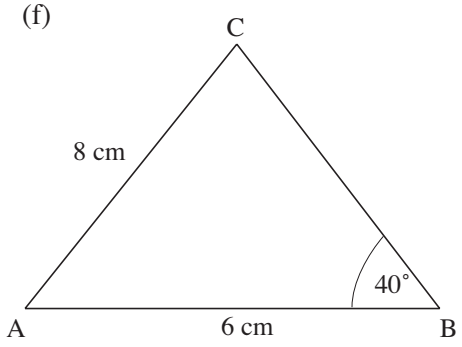
What is the size of the angle ABC?

(e)



How long is the side AC?

(f)



How long is the side BC?

3. An isosceles triangle ABC has $AB = 6$ cm and angles ABC and CAB each equal to $50°$. Find the lengths of the other sides of the triangle.

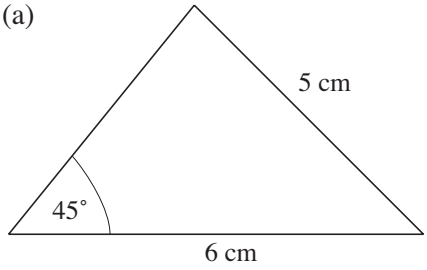
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4. An isosceles triangle has 2 sides of length 8 cm and one side of length 4 cm. Find the sizes of all the angles in the triangle.

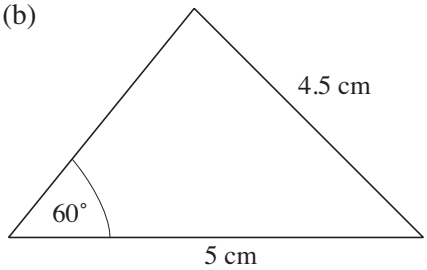
5. Draw an equilateral triangle with sides of length 5 cm.

6. For each rough sketch shown below, draw two possible triangles.

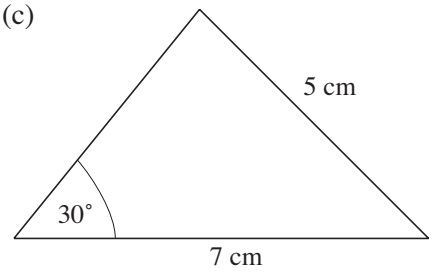
(a)



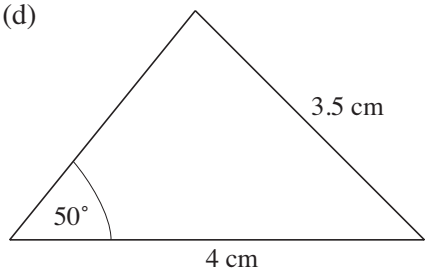
(b)



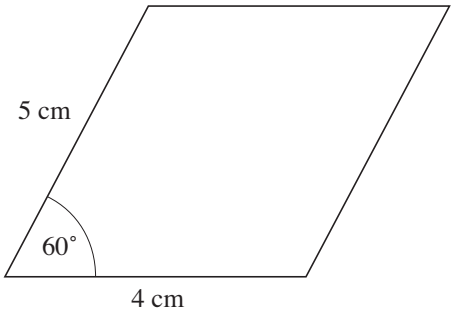
(c)



(d)

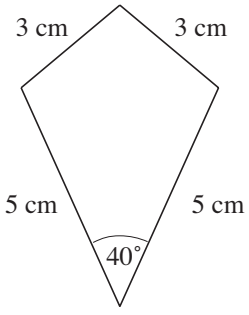


7. (a) Draw accurately the parallelogram shown below.



(b) Measure the two diagonals of the parallelogram.

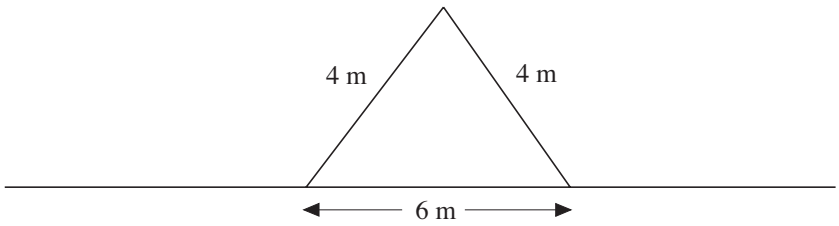
8. (a) Draw the kite shown in the rough sketch opposite.



(b) Check that the diagonals of the kite are at right angles.

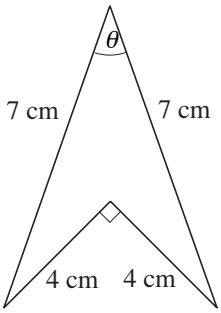
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9. A pile of sand has the shape shown below. Using an accurate diagram, find its height.



10. Draw accurately the shape shown opposite.

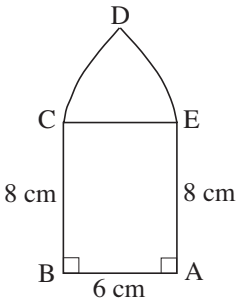
Find the size of the angle marked θ.



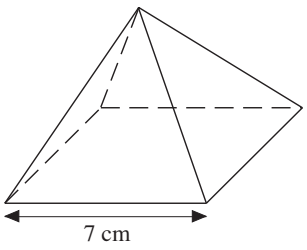
11. The sketch shows the design for a church window.

CB and EA are perpendicular to BA.
CD is part of a circle, centre E.
DE is part of a circle, centre C.

Using a ruler and compasses draw the design accurately.

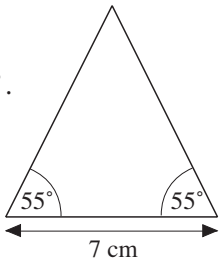


12. John is required to construct a pyramid with a square base, as shown below.



(a) Each sloping face is a triangle with base angles of $55°$.

Construct one of these triangles accurately and to full size.



(b) Construct the square base of the pyramid accurately and to full size.

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13. A rectangle has sides of 8 cm and 5 cm.

    (a) Calculate the perimeter of the rectangle.

    (b) Construct the rectangle accurately.

14. Construct a rhombus ABCD with the line $AB = 4\text{ cm}$ as base and with $B\hat{A}D = 50°$.

15. Using ruler and compasses ONLY:

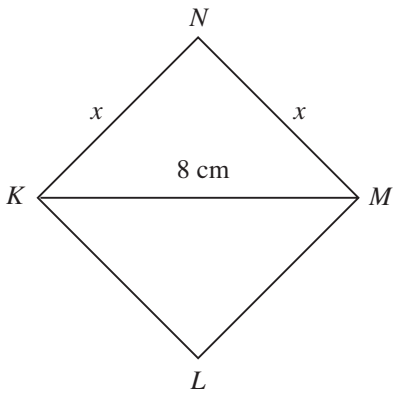
    (a) Construct the triangle *FGH* with $FG = 7.5\text{ cm}$, angle $FGH = 120°$ and angle $GFH = 30°$.

    (b) Locate on *FG*, the point *M*, the midpoint of *FG*.

    Show all construction lines.

    Measure and state the size of angle *GMH*.

16. (a)



The diagram above, **not drawn to scale**, shows a square *KLMN*, where $KM = 8\text{ cm}$ and $KN = MN = x\text{ cm}$.

Show that $x^2 = 32$.

(b) (i) **Using ruler and compasses only**,

a) draw the diagonal $KM = 8\text{ cm}$

b) construct the perpendicular bisector of *KM*.

(ii) Hence, draw the square *KLMN*.

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H.3.3 Straight Edge and Compasses Constructions

In this section we consider how to draw some shapes and specific angles using only a straight edge and compasses.

We begin by drawing the key constructions.

Construction of a Perpendicular at a Point on a Line

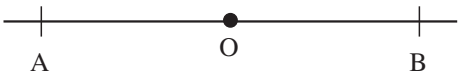
Worked Example 1

Construct a perpendicular at a point, O, on a line.

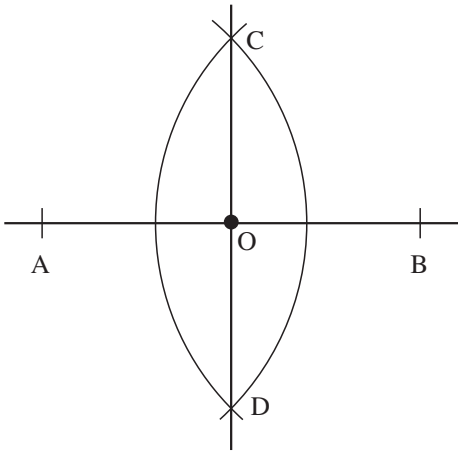


Solution

Use compasses to mark points A and B on the line that are at equal distances from the given point O.



Draw arcs centred at A and B. These two arcs intersect at C and D. Then the required perpendicular line can be drawn through the points C and D.



Construction of the Perpendicular Bisector of a Line Segment

Worked Example 2

Construct the perpendicular bisector of the line AB.

Then label the midpoint of AB, M.

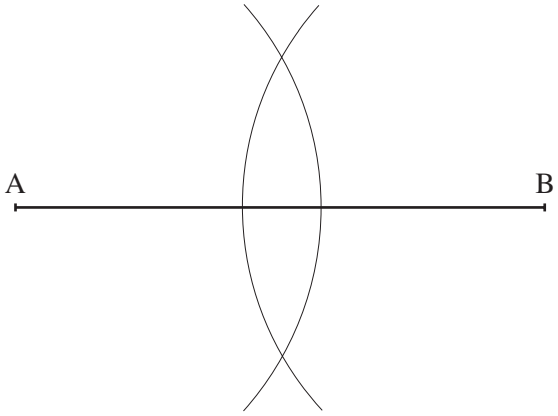


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Solution

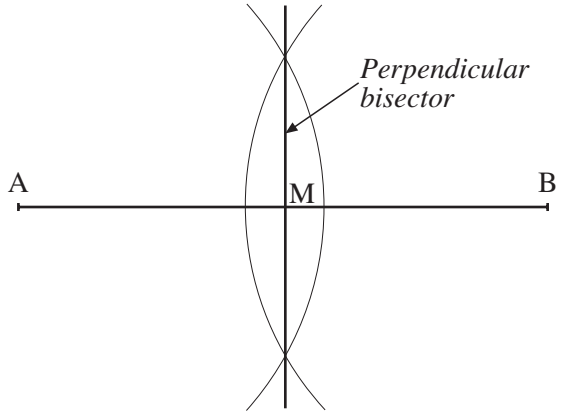
There are many lines that cut AB exactly in half. We have to construct the one that is perpendicular to AB.

We begin by drawing arcs of equal radius, centred on the points A and B, as shown in the diagram.



The radius of these arcs should be roughly $\frac{2}{3}$ to $\frac{3}{4}$ of the length AB.

Then draw a line through the intersection points of the two arcs. This line is the perpendicular bisector of AB.



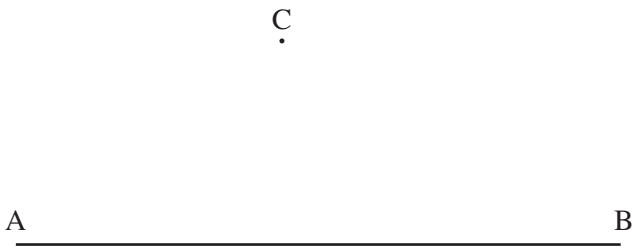
The point where the perpendicular bisector intersects AB can then be labelled M. M is the midpoint of AB.

Construction of the Perpendicular to a Line from a Point not on the Line

Worked Example 3

The diagram shows the line AB and the point C.

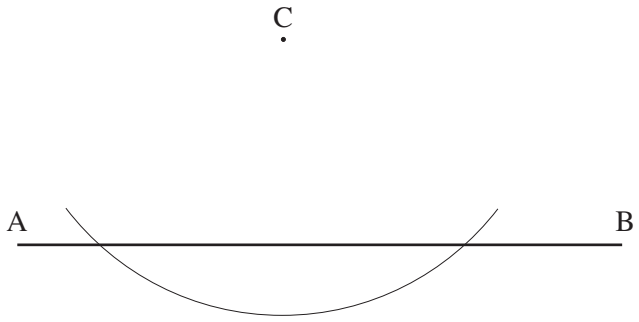
Draw a line through C that is perpendicular to AB.



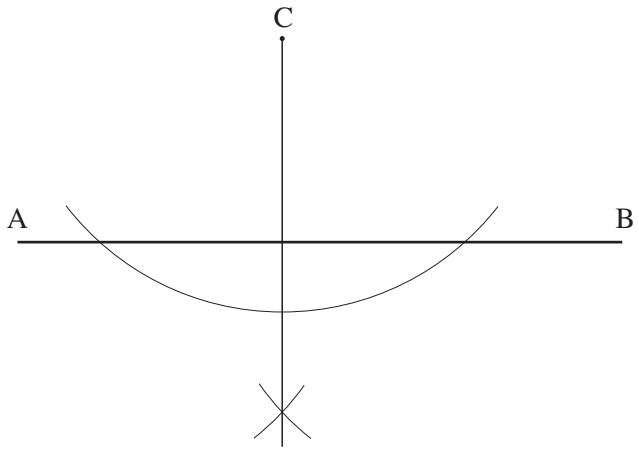
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Solution

Using C as the centre, draw an arc as shown.



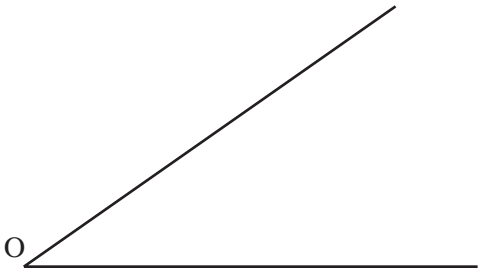
Then using the intersection points of this arc with the line AB as centres, draw two further arcs with radii of equal length. The perpendicular line can then be drawn from C through the point where these two new arcs cross.



Construction of the Bisector of an Angle

Worked Example 4

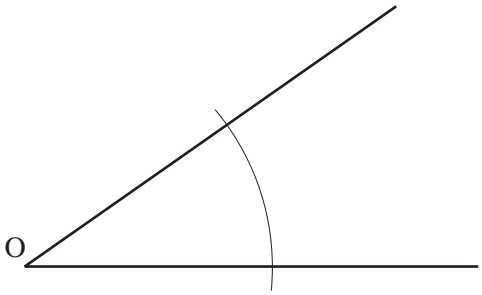
Bisect this angle.



Solution

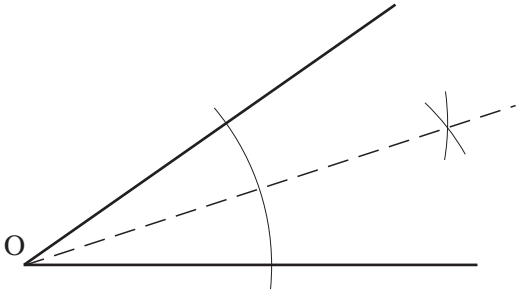
To bisect an angle you need to draw a line that cuts the angle in half.

First draw an arc using O as the centre.



Then draw two further arcs of equal radius, using the points where the arc intersects the lines as the centres.

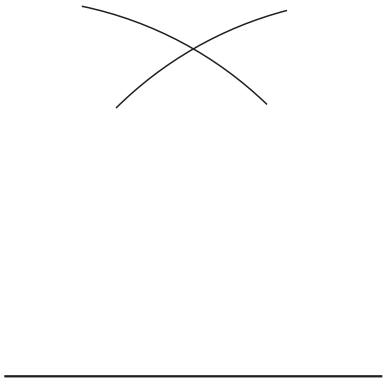
The bisector can then be drawn from O through the point where these two new arcs cross.



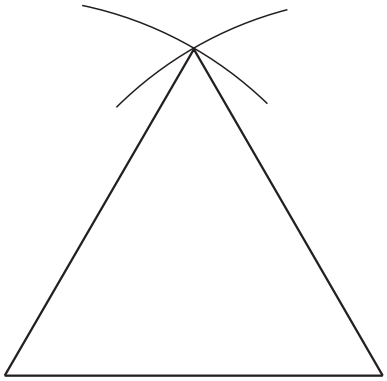
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Uses of these constructions

Construction 1: *To draw an equilateral triangle*

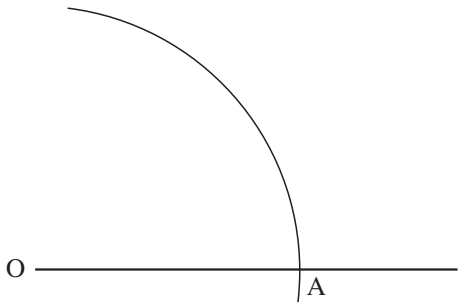


Draw one side. Then draw two arcs, one from each end of this side, with radius the same as the length of the first side of the triangle.

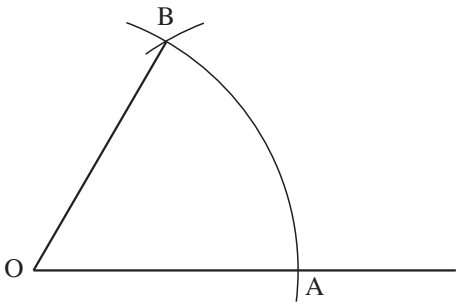


Join the ends of the line to the intersection of the arcs to complete the triangle

Construction 2: *To draw an angle of* $60°$



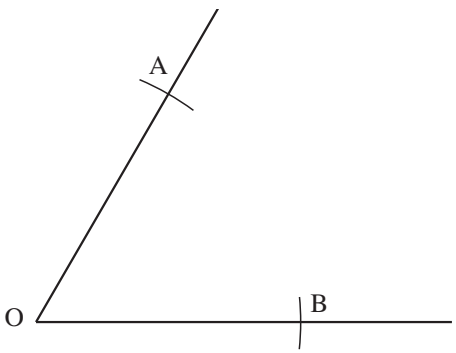
If the angle of $60°$ is to be drawn at O, draw a line through O that will form one side of the angle. Then draw an arc, with the compass point at O, crossing the drawn side at A, as shown above.

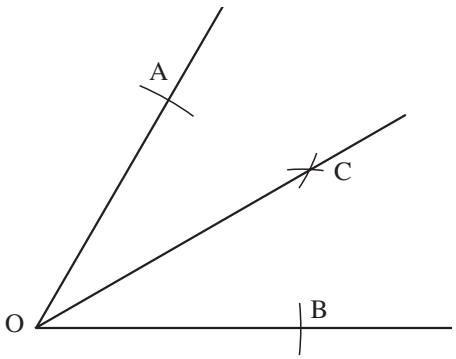


Draw a second arc, of the same radius, with the compass point at A, intersecting the first arc at B. Then join O to B to complete the angle

Construction 3: *To draw an angle of* $30°$

First draw an angle of $60°$ using the previous construction, then bisect that angle using the method described in Worked Example 4 earlier in this section.

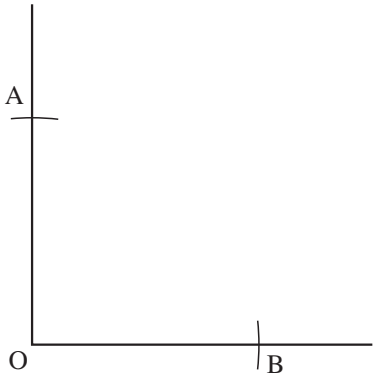


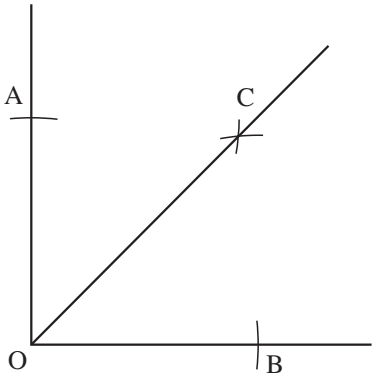


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Construction 4: *To draw an angle of* $45°$

Start by constructing a $90°$ angle using one of the methods described in Worked Examples 1, 2 and 3. Then bisect that angle using the method described earlier in Worked Example 4.





Worked Example 5

Construct a triangle ABC with AB = 8 cm, the angle BAC = $60°$ and ABC = $45°$.

Then measure

(a) BC

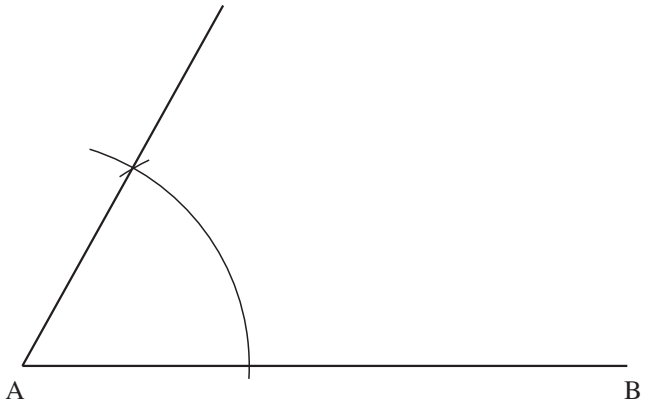
(b) AC

(c) the angle ACB.

Solution

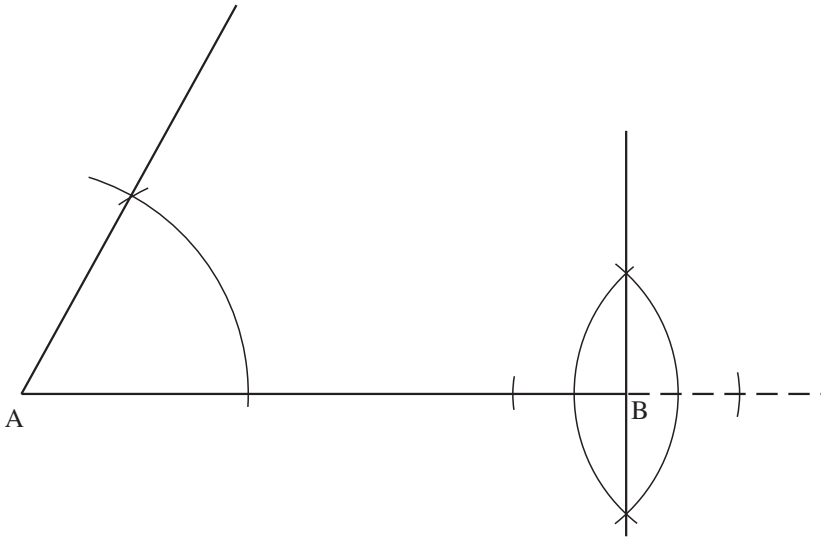
First draw a line of length 8 cm and label it AB.

Then construct an angle of $60°$ at the end A as shown below (see construction 2 above).

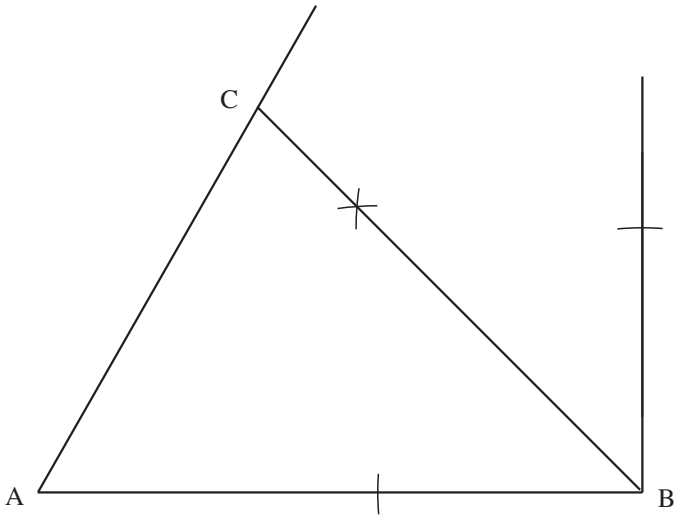


Extend the line AB and draw the line through B perpendicular to AB, using the method described in Worked Example 1.

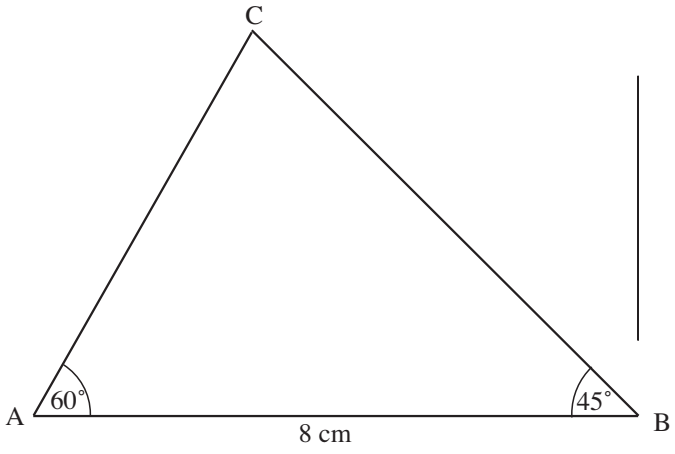
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Using the method described in Worked Example 4, this right angle can now be bisected to give the $45°$ angle that is required. This is shown in the following diagram and completes the triangle.



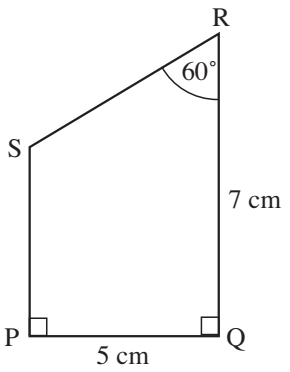
The triangle is a shown below:



(a) BC = 7.2 cm (b) AC = 5.9 cm (c) $\angle ACB = 75°$

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Worked Example 6



Use a ruler and compasses to construct the shape PQRS as shown opposite.

Then measure

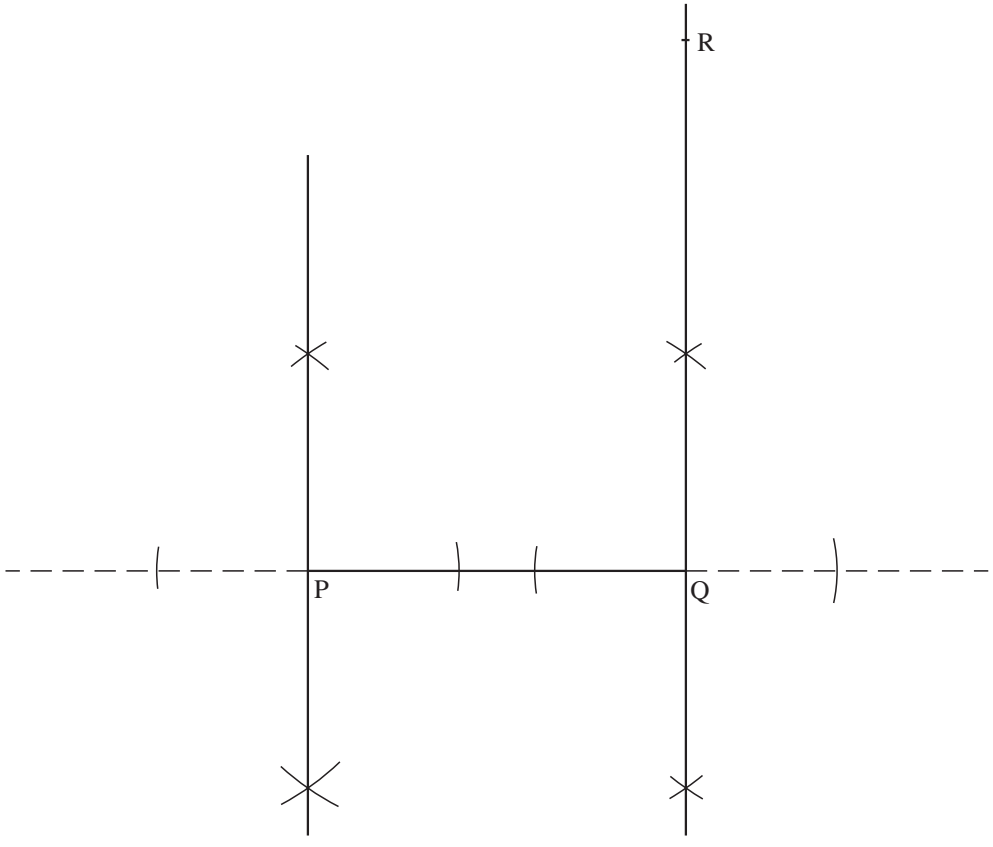
(a) RS

(b) PS

(c) the angle PSR.

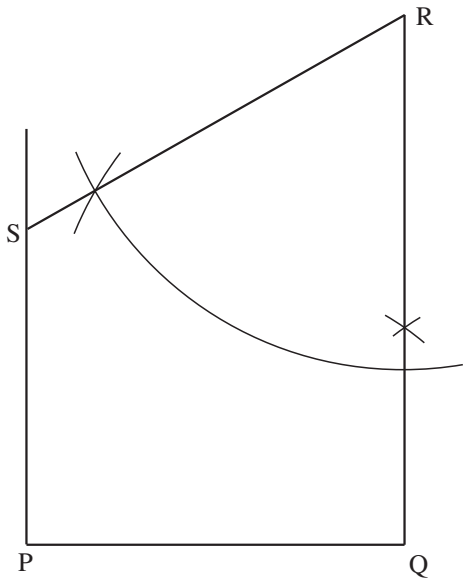
Solution

First draw the line PQ, with length 5 cm, and then construct perpendicular lines through P and Q using the method described in Worked Example 1. Measure 7 cm along the perpendicular through Q and mark in the point R.

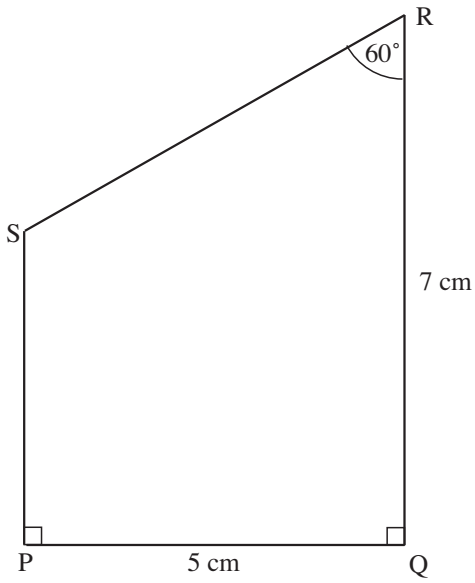


Now, using the method described in Construction 2, construct a $60°$ angle at R to complete the shape.

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The shape PQRS is shown below:



(a) RS = 5.8 cm

(b) PS = 4.1 cm

(c) $\angle PSR = 120°$

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Exercises

1. Construct an angle of $135°$. (*Hint*: $135° = 90° + 45°$)

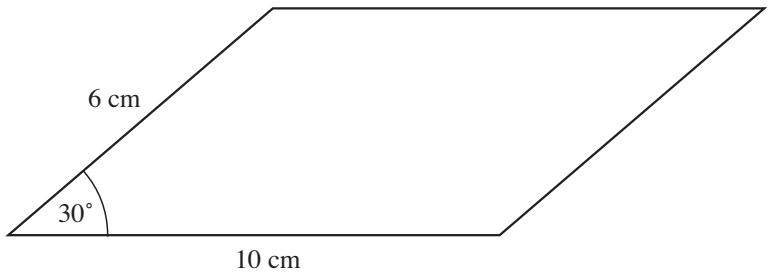
2. Construct an angle of $15°$.

3. Construct an angle of $75°$.

4. A triangle ABC is such that AB = 10 cm, $\angle ABC = 30°$ and $\angle BAC = 60°$.
   (a) Draw the triangle.
   (b) Measure AC and BC.
   (c) Measure $\angle ACB$.

5. Draw triangle LMN with LM = 9 cm, $\angle LMN = 30°$ and LN = 7 cm.
   Show that there are two possible solutions.

6. Use a ruler and compasses to construct this parallelogram.



7. Take any triangle ABC. Set your compasses to length BC and draw an arc centred on A. Now set your compasses to length AB and draw an arc centred on C. Label the point of intersection, D. Join D to A and C. Explain why this construction produces the line through A parallel to BC and the line through C parallel to BA.

8. Take any triangle XYZ. Construct the perpendicular bisectors of XY, YZ and ZX. Show that these meet at a single point, W. Draw the circle, centre W, radius WX. Show that this circle passes through all three vertices of the triangle. (It is called the *circumscribed circle* of ΔXYZ.)

9. Take any triangle JKL. Construct the angle bisectors of $\angle JKL$, $\angle KLJ$ and $\angle LJK$. Show that these meet at a single point, I. Draw the circle, centre I, such that this circle just touches all three sides of the triangle. (It is called the *inscribed circle* of ΔJKL.)

10. Using straight edge and compasses:
    (a) construct a regular hexagon, (b) construct a regular octagon.

11. Take any triangle PQR. Construct the perpendicular from P to QR meeting it at S, the perpendicular from Q to PR meeting it at T, and the perpendicular from R to PQ meeting it at U. Show that the lines PS, QT and RU all meet at a single point. (It is called the *orthocentre* of ΔPQR.)

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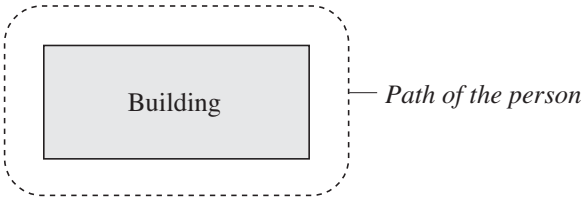
12. Take any triangle ABC. Construct the perpendicular bisector of BC to obtain the midpoint F of BC. Construct the midpoint G of CA, and the midpoint H of AB, in the same way. Join AF, BG and CH. Show that these meet at a single point, M. Show, by measuring, that

$$AM = \frac{2}{3}AF, \qquad BM = \frac{2}{3}BG \qquad CM = \frac{2}{3}CH$$

(M is called the *centroid* of ΔABC. If you cut ΔABC out of card then M would be the centre of mass of ΔABC.)

H.3.4 Construction of Loci

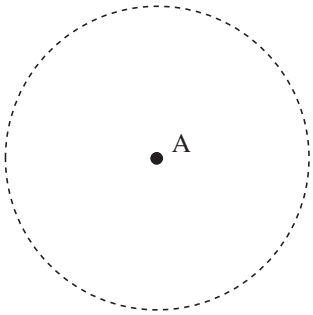
When a person moves so that they always satisfy a certain condition, their possible path is called a *locus*. For example, consider the path of a person who walks around a building, always keeping the same distance away from the building.



The dotted line in the diagram shows the path taken – i.e. the *locus*.

Worked Example 1

Draw the locus of a point which is always a constant distance from another point.



Solution

The fixed point is marked A.
The locus is a circle around the fixed point.

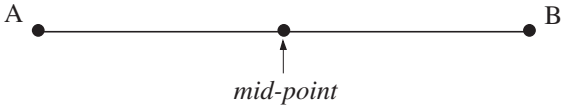
Worked Example 2

Draw the locus of a point which is always the same distance from A as it is from B.



Solution

The locus of the line AB will be the same distance from both points.

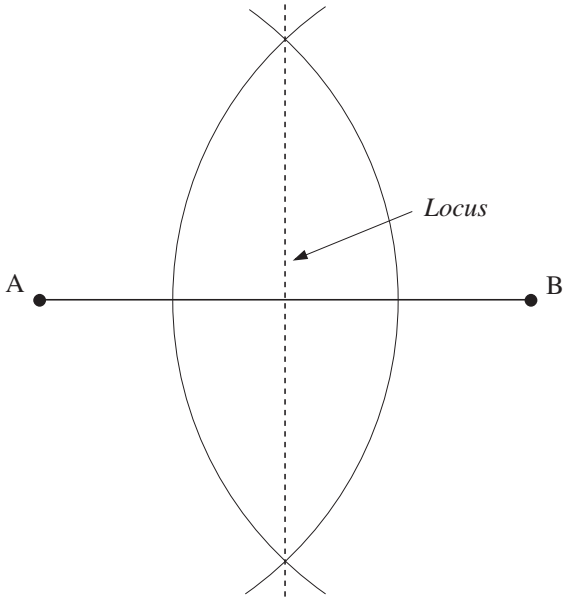


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However, any point on a line perpendicular to AB and passing through the mid-point of AB will also be the same distance from A and B.

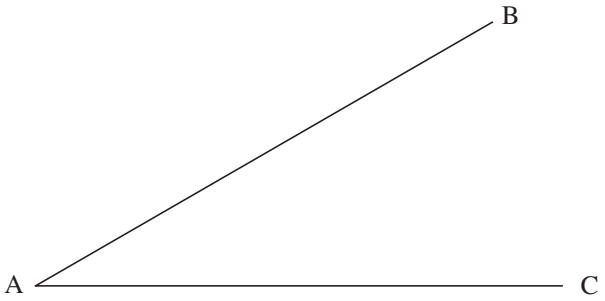
The diagram shows how to construct this line.

This line is called the *perpendicular bisector* of AB.



Worked Example 2

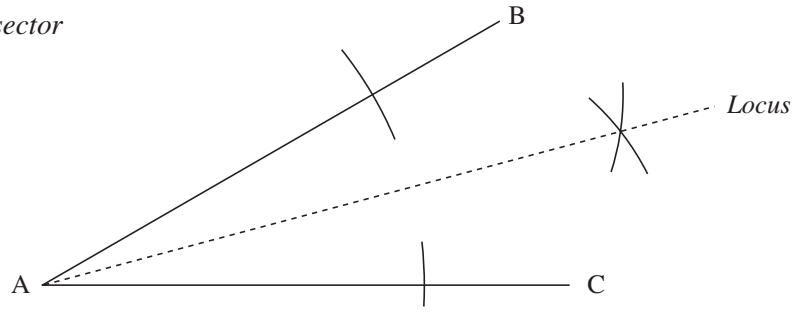
Draw the locus of a point that is the same distance from the lines AB and AC shown in the diagram below.



Solution

The locus will be a line which divides the $B\hat{A}C$ into two equal angles. The diagram below shows how to construct this locus.

This line is called the *bisector* of $B\hat{A}C$.



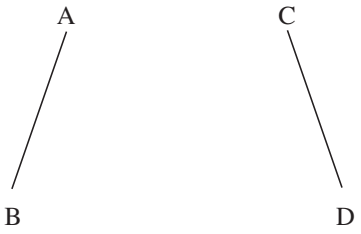
Exercises

1. Draw the locus of a point which is always a distance of 4 cm from a fixed point A.

2. The line AB is 4 cm long. Draw the locus of a point which is always 2 cm from the line AB.

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3. Copy the diagram and draw in the locus of a point which is the same distance from the line AB as it is from CD.

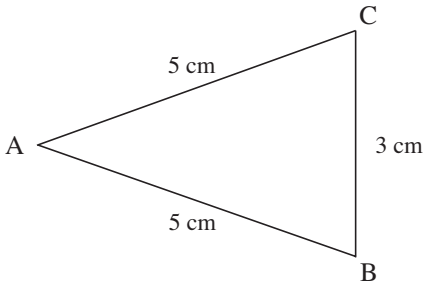


4. (a) Draw an equilateral triangle with sides of length 5 cm.

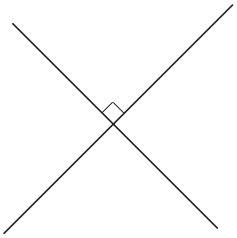
   (b) Draw the locus of a point that is always 1 cm from the sides of the triangle.

5. Copy the triangle opposite.

   Draw the locus of a point which is the same distance from AB as it is from AC.



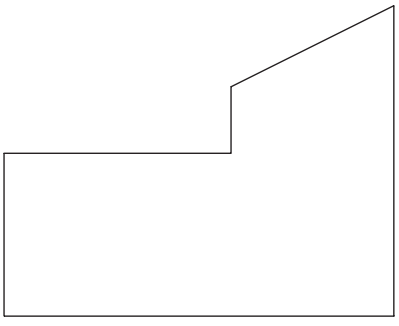
6. Draw the locus of a point that is the same distance from both lines shown in the diagram below.



7. (a) Draw 2 parallel lines.

   (b) Draw the locus of a point which is the same distance from both lines.

8. The diagram below shows the boundary fence of a high security army base.



   (a) Make a copy of this diagram.

   (b) A security patrol walks round the *outside* of the base, keeping a constant distance from the fence. Draw the locus of the patrol.

   (c) A second patrol walks *inside* the fence, keeping a constant distance from the fence. Draw the locus of this patrol.

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9. The points A and B are 3 cm apart. Draw the locus of a point that is twice as far from A as it is from B.

10. A ladder leans against a wall, so that it is almost vertical. It slides until it is flat on the ground. Draw the locus of the mid-point of the ladder.

11. The points A and B are 4 cm apart.

    (a) Draw the possible positions of the point, P, if

    (i) $AP = 4$ cm and $BP = 1$ cm

    (ii) $AP = 3$ cm and $BP = 2$ cm

    (iii) $AP = 2$ cm and $BP = 3$ cm

    (iv) $AP = 1$ cm and $BP = 4$ cm

    (b) Draw the locus of the point P, if $AP + BP = 5$ cm.

    (c) Draw the locus of the point P, if $AP + BP = 6$ cm.

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